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CONVERGENCE ANALYSIS OF VOLTERRA SERIES RESPONSE OF NONLINEAR SYSTEMS SUBJECTED TO HARMONIC EXCITATION

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Volterra series provides a strong platform for non-linear analysis and higher order frequency response functions. However, limited convergence is an inherent difficulty associated with the series and needs to be addressed rigorously, prior to its application to a physical system. The power series representation of the response of non-linear systems, subjected to harmonic excitation is investigated in this study. The problem of convergence is addressed in terms of the convergence of individual frequency harmonics of the non-linear response. Though the procedure is applicable to general polynomial form non-linearity, it is illustrated for a Duffing oscillator subjected to harmonic excitation. A general and structured series expression is obtained for amplitudes of all the response harmonics and convergence is investigated in terms of a non-dimensional non-linear parameter. Critical values of this parameter, representing the upper limit of excitation level for the convergence, are defined for a wide range of excitation frequencies. Zones of convergence and divergence of the response series are presented graphically, for a range of the non-dimensional non-linear parameter and the number of terms included in the approximation of a response harmonic. An algorithm based on ratio test is presented to compute the critical value of the non-dimensional non-linear parameter. Results obtained from the suggested algorithm are found to be in close agreement with the exact values. The method gives better results compared to previous methods and has wider application in terms of excitation frequency. The procedure is also investigated for a two-degree-of-freedom system.

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1. INTRODUCTION

The functional form representation of input–output relationship through Volterra series [1, 2], provides a structured and convenient mathematical platform for the study of non-linear systems. It employs multidimensional kernels, which upon convolution with the applied excitation, express the response in the form of a power series. The Volterra series, being an infinite power series with memory, however suffers from the problem of limited convergence and has been applied in relatively fewer situations [3–5]. It is required to be truncated to a finite number of terms, in all practical computations and may lead to large errors. The problem is sought to be overcome, through formation of an orthogonal set of Wiener kernels [6] from Volterra functionals, for white Gaussian inputs. Applications based on Wiener kernels can be found in [7–11]. Since the Wiener series analysis is carried out for Gaussian white excitation, its practical application involves statistical errors and limitations. Moreover, Wiener kernels are excitation level dependent and need to be converted to Volterra kernels, for their translation into system characteristics [12]. Studies [13, 14] on the convergence aspects of Volterra series, have been relatively few. Sandberg

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[15] has shown that a truncated Volterra series provides a uniform approximation to the infinite Volterra series on a ball of bounded input for a large class of systems. Analysis and identification of non-linear systems by harmonic excitation using Volterra series theory have been reported in references [8, 16, 17]. Stiffness non-linearity is commonly observed under large excitation force. Tomlinson and Manson [18] studied the convergence of first order FRF of a Duffing oscillator under harmonic excitation and presented a simple formula for determining the upper limit of excitation level. However, the formula gives accurate results only at resonant frequency and for a two-term Volterra approximation. At driving frequencies away from natural frequency and for higher order Volterra approximation, results deviate significantly from the exact ones. In this paper, convergence of the response harmonics for a Duffing oscillator under harmonic excitation is studied in terms of a non-dimensional non-linear parameter. A convergence criterion, based on the number of terms in the approximation of a response harmonic, is suggested. Critical values of the non-dimensional non-linear parameter, are defined for convergence. Numerical computation is carried out, using an algorithm based on ratio test. Zones of convergence and divergence are specified in terms of the non-linear parameter and number of terms in the series. The convergence criteria improves upon that by Tomlinson by defining a critical value of the non-dimensional non-linear parameter as a function of excitation level, excitation frequency and the number of terms in the Volterra series approximation. Convergence of two-degree-of-freedom systems subjected to harmonic excitation is also discussed.

2. VOLTERRA SERIES RESPONSE REPRESENTATION

A single-degree-of-freedom system, with general polynomial form of non-linearity is considered

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) + g[x(t), \dot{x}(t)] = f(t). \tag{1}$$

$f(t)$ is the harmonic excitation,

$$f(t) = A\cos\omega t = \frac{A}{2}\,\mathrm{e}^{\mathrm{j}\omega t} + \frac{A}{2}\,\mathrm{e}^{-\mathrm{j}\omega t} \tag{2}$$

and the non-linear term $g[x(t), \dot{x}(t)]$ is expressed in general polynomial form as

$$g[x(t), \dot{x}(t)] = k_2 x^2(t) + k_3 x^3(t) + \cdots + c_2\dot{x}^2(t) + c_3\dot{x}^3(t) + \cdots. \tag{3}$$

Using Volterra series representation [1] of the response, $x(t)$ is expressed as

$$x(t) = \sum_{n=1}^{\infty} x_n(t) \tag{4}$$

with the nth order response component being given by

$$x_n(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} h_n(\tau_1, \tau_2, \ldots, \tau_n) f(t-\tau_1) f(t-\tau_2)\ldots f(t-\tau_n)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2\ldots\mathrm{d}\tau_n \tag{5}$$

$h_n(\tau_1, \tau_2, \ldots, \tau_n)$ is nth order Volterra kernel and its Fourier transform [2] provides the nth order frequency response function as

$$H_n(\omega_1, \omega_2, \ldots, \omega_n) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1, \tau_2, \ldots, \tau_n) \prod_{i=1}^{n} \mathrm{e}^{-\mathrm{j}\omega_i \tau_i}\, \mathrm{d}\tau_1\, \mathrm{d}\tau_2 \ldots \mathrm{d}\tau_n. \quad (6)$$

Employing the above, the individual response components (equation (5)), can be expressed, after some algebra as

$$x_n(t) = \left(\frac{A}{2}\right)^n \sum_{p+q=n} {}^nC_q H_n^{p,q}(\omega) \mathrm{e}^{\mathrm{j}\omega_{p,q} t}, \quad 0 \leqslant p \leqslant n;\ 0 \leqslant q \leqslant n, \quad (7)$$

where the following brief notations have been used:

$$H_n^{p,q}(\omega) = H_n\Bigg(\underbrace{\omega, \ldots, \omega}_{p\ times}, \underbrace{-\omega, \ldots, -\omega}_{q\ times}\Bigg), \quad \omega_{p,q} = (p-q)\,\omega. \quad (8)$$

The total response of the system then becomes

$$x(t) = \sum_{n=1}^{\infty} \left(\frac{A}{2}\right)^n \sum_{p+q=n} {}^nC_q H_n^{p,q}(\omega) \mathrm{e}^{\mathrm{j}\omega_{p,q} t}. \quad (9)$$

Combinations of different p and q result in various response harmonics at frequencies $\omega_{p,q}$ and the response can be written in terms of its harmonics as

$$x(t) = X_0 + |X(\omega)| \cos(\omega t + \phi_1) + |X(2\omega)| \cos(2\omega t + \phi_2) + |X(3\omega)| \cos(3\omega t + \phi_3) + \cdots, \quad (10)$$

where

$$X_0(t) = \sum_{n=1}^{\infty} \left(\frac{A}{2}\right)^{2n} {}^{2n}C_q H_{2n}^{n,n}(\omega)$$

$$X(n\omega) = 2 \sum_{i=1}^{\infty} \left(\frac{A}{2}\right)^{n+2i-2} {}^{n+2i-2}C_{i-1} H_{n+2i-2}^{n+i-1,i-1}(\omega) \quad \text{for } n = 1, 2, \ldots$$

$$\phi_n = \angle X(n\omega). \quad (11)$$

3. CONVERGENCE FOR A DUFFING OSCILLATOR

For a Duffing oscillator, accounting for stiffness non-linearity alone, and excited by the harmonic force of equation (2), the governing equation is

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) + k_3 x^3(t) = \frac{A}{2}\mathrm{e}^{\mathrm{j}\omega t} + \frac{A}{2}\mathrm{e}^{-\mathrm{j}\omega t}. \quad (12)$$

Defining

$$\tau = \omega_n t, \quad \omega_n = \sqrt{k/m}, \quad \varsigma = c/2m\omega_n, \quad z = x/X_{st}$$

$$X_{st} = A/k, \quad \lambda = k_3 A^2/k^3, \quad r = \omega/\omega_n,$$

equation (12) can be rewritten in non-dimensional form as

$$z''(\tau) + 2\varsigma z'(\tau) + z(\tau) + \lambda z^3(\tau) = \frac{e^{jr\tau}}{2} + \frac{e^{-jr\tau}}{2}, \tag{13}$$

where ($'$) denotes differentiation with respect to τ.

Using expression (9), similar series form for the non-dimensional response $z(\tau)$ would be

$$z(\tau) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \sum_{p+q=n} {}^nC_q H_n^{p,q}(r) e^{jr_{p,q}\tau}, \tag{14}$$

where $H_n^{p,q}(r)$ and $r_{p,q}$ are defined similar to $H_n^{p,q}(\omega)$ and $\omega_{p,q}$ respectively.

Substitution of the Volterra series of equation (14), in the equation of motion (13) yields

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \sum_{p+q=n} {}^nC_q H_n^{p,q}(r) e^{jr_{p,q}\tau} [-r_{p,q}^2 + 1 + j2\varsigma r_{p,q}]$$
$$+ \lambda \left[\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \sum_{p+q=n} {}^nC_q H_n^{p,q}(r) e^{jr_{p,q}\tau} \right]^3 = \frac{1}{2} e^{jr\tau} + \frac{1}{2} e^{-jr\tau}. \tag{15}$$

Applying the method of harmonic probing [8] and equating the coefficients of $(1/2)^n e^{jr_{p,q}\tau}$, one obtains

$$H_1(r) = 1/(-r^2 + 1 + j2\varsigma r) \quad \text{for } n = 1 \tag{16a}$$

and

$$H_n^{p,q}(r) = -\frac{\lambda}{{}^nC_q} H_1(r_{p,q}) \sum_{\substack{p_i + q_i = n_i \\ n_1 + n_2 + n_3 = n}} [{}^{n_1}C_{q_1} H_{n_1}^{p_1,q_1}(r)] * [{}^{n_2}C_{q_2} H_{n_2}^{p_2,q_2}(r)]$$
$$* [{}^{n_3}C_{q_3} H_{n_3}^{p_3,q_3}(r)] \quad \text{for } n > 1. \tag{16b}$$

Thus, a higher order kernel transform $H_n^{p,q}(r)$ can be reduced into factors of lower order kernel transforms. In equation (16b), the summation is to be carried out over all sets of n_i satisfying $n_1, n_2, n_3 > 0$ and $n_1 + n_2 + n_3 = n$. This criterion is not satisfied for $n = 2$, and therefore $H_2^{p,q}(r) = 0$. Similarly, the higher even order kernel transforms, $H_4^{p,q}(r)$, $H_6^{p,q}(r)$ etc., vanish for a Duffing oscillator, confining the response to the odd harmonics alone, as

$$z(\tau) = |Z(r)| \cos(r\tau + \psi_1) + |Z(3r)| \cos(3r\tau + \psi_3) + |Z(5r)| \cos(5r\tau + \psi_5) + \cdots, \tag{17}$$

where $\psi_n = \angle Z(nr)$.

Each of the response harmonics $Z(nr)$ can be seen to be comprised of an infinite power series

$$Z(nr) = \sum_{i=1}^{\infty} \sigma_i(nr), \tag{18}$$

where, using series expression (11), the individual terms, $\sigma_i(nr)$, can be written as

$$\sigma_i(nr) = 2 \left(\frac{1}{2}\right)^{n+2i-2} {}^{n+2i-2}C_{i-1} H_{n+2i-2}^{n+i-1,i-1}(r). \tag{19}$$

The practical estimation of a harmonic can incorporate only a finite number of terms of the infinite series (18). The convergence of the response harmonic then would be specific to the number of terms included in the analysis. Confining the power series to include a finite number of terms, k, the approximation for the nth harmonic can be denoted as

$$ {}_kZ(nr) = \sum_{i=1}^{k} \sigma_i(nr) \tag{20}$$

and the relative error between the above approximation and the exact amplitude of the harmonic is

$$ {}^n_k e = |[Z(nr) - {}_kZ(nr)]/Z(nr)|. \tag{21}$$

4. NUMERICAL ILLUSTRATION

The response of the governing equation (13) is computed numerically in non-dimensional form, through fourth-order Runge–Kutta method. Figures 1(a) and 1(b) typically show the non-dimensional response $z(\tau)$ and amplitudes of dominant response harmonics for $r = 0{\cdot}35$, $\lambda = 0{\cdot}04$. Amplitudes of various response harmonics, $Z(r), Z(3r)$, etc., are obtained through Fourier analysis or harmonic filtering of the response $z(\tau)$. These are termed as the "exact" response harmonics.

The Volterra series response is synthesized by considering the series in equation (18) up to k number of terms. Each series term, $\sigma_i(nr)$, $i = 1, 2, \ldots, k$, is computed from equation (19), in which the higher order kernel transforms $H^{n+i-1,\, i-1}_{n+2i-2}(r)$ are obtained by step-by-step reduction into lower order transforms using equation (16b). The series form of response harmonic amplitude $Z(r)$, following equations (18), (19) and (16b) becomes

$$\begin{aligned} Z(r) &= H_1(r) + \frac{3}{4} H_3(r, r, -r) + \frac{5}{8} H_5(r, r, r, -r, -r) + \cdots \\ &= H_1(r) - \frac{3}{4}\lambda H_1^3(r) H_1(-r) + \lambda^2 \begin{bmatrix} \frac{9}{16} H_1^4(r) H_1^3(-r) + \frac{9}{8} H_1^5(r) H_1^2(-r) \\ + \frac{3}{16} H_1^4(r) H_1^2(-r) H_1(3r) \end{bmatrix} + \cdots. \end{aligned} \tag{22}$$

Similarly series form of response harmonic amplitude $Z(3r)$ becomes

$$\begin{aligned} Z(3r) &= \frac{1}{4} H_3(r, r, r) + \frac{5}{16} H_5(r, r, r, r, -r) + \cdots \\ &= -\frac{\lambda}{4} H_1^3(r) H_1(3r) + \lambda^2 \begin{bmatrix} \frac{9}{16} H_1^5(r) H_1(-r) H_1(3r) \\ + \frac{3}{8} H_1^4(r) H_1(-r) H_1^2(3r) \end{bmatrix} + \cdots. \end{aligned} \tag{23}$$

Using the above equations, a three-term approximation of $Z(r)$ and two-term approximation of $Z(3r)$ can be computed directly employing the expression for $H_1(r)$, equation (16a). In addition, the overall Volterra series response $z(\tau)$ can be constructed from

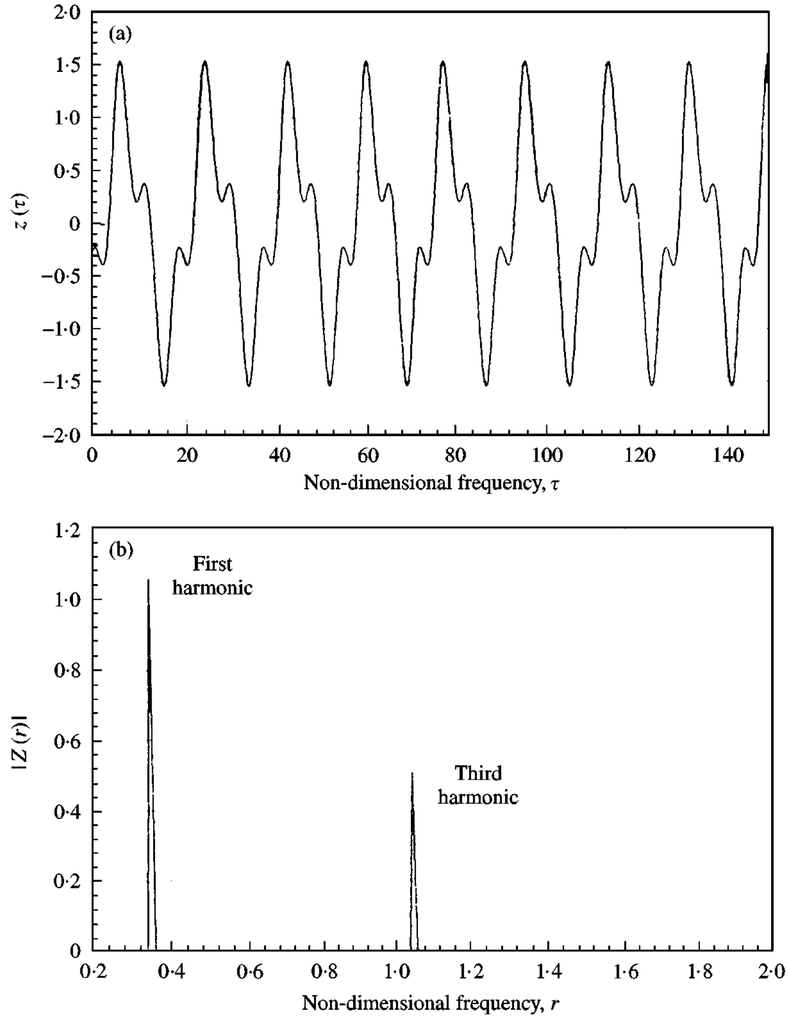

Figure 1. Typical non-dimensional response $z(\tau)$ and its dominant harmonic amplitudes, ($\lambda = 0{\cdot}04$, excitation frequency at $r = 0{\cdot}35$): (a) response $z(\tau)$; (b) response spectrum $|Z(r)|$.

the response harmonic amplitudes as

$$z(\tau) = z_1(\tau) + z_3(\tau) + \cdots, \tag{24}$$

where

$$z_1(\tau) = \sigma_1(r)\,\mathrm{e}^{\mathrm{j}r\tau} + \sigma_1^*(r)\,\mathrm{e}^{-\mathrm{j}r\tau}$$

$$z_3(\tau) = \sigma_2(r)\,\mathrm{e}^{\mathrm{j}r\tau} + \sigma_2^*(r)\,\mathrm{e}^{-\mathrm{j}r\tau} + \sigma_1(3r)\,\mathrm{e}^{\mathrm{j}3r\tau} + \sigma_1^*(3r)\,\mathrm{e}^{-\mathrm{j}3r\tau}, \text{ etc.}$$

$\sigma_1^*(r)$, $\sigma_2^*(r)$ and $\sigma_1^*(3r)$ are respectively the complex conjugates of $\sigma_1(r)$, $\sigma_2(r)$ and $\sigma_1(3r)$. Figures 2(a) and 2(b) show the phase-plane trajectories of response synthesized from Volterra series approximation and response obtained from numerical integration for a typical non-dimensional frequency, $r = 0{\cdot}6$. Convergence of Volterra series response can

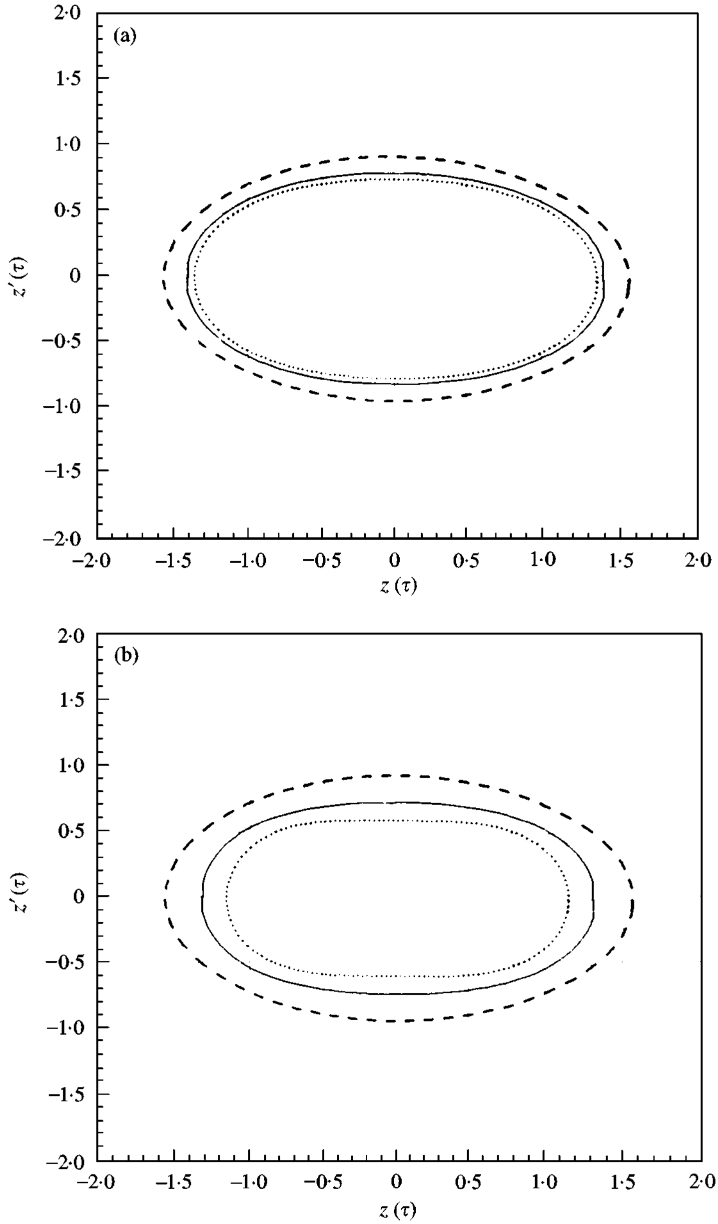

Figure 2. Phase-plane trajectories of response obtained from Volterra series and from numerical integration: (a) $r = 0{\cdot}6$ and $\lambda = 0{\cdot}05$; (b) $r = 0{\cdot}6$ and $\lambda = 0{\cdot}1$. ——, rk-4; ---- 1-term Volterra; ····, 2-term Volterra.

be graphically represented by the proximity of the trajectory to that of response obtained from numerical integration. It can be seen that the phase plot of a two-term Volterra series is closer to the exact one than the single-term Volterra series, indicating a converging trend. The phase plots also highlight the alternating nature of the Volterra series in this case and show that better convergence is obtained with lower values of λ. However, the phase-plane comparison can be used for the convergence study of total response only.

Convergence of individual response harmonic amplitudes is analyzed by computing the relative errors ${}^{n}_{k}e$ (equation (21)), for various values of non-dimensional non-linear parameter λ. The variation of the relative errors, ${}^{1}_{k}e$ and ${}^{3}_{k}e$, between the "exact" and k-term approximations of the harmonics for various values of k and non-dimensional non-linear parameter, λ, have been plotted in Figures 3(a) and 3(b) (for non-dimensional excitation frequency, $r = 0{\cdot}6$). The errors can be seen to decrease up to a certain number of terms in the approximation, beyond which they display an increasing trend. The number of terms up to which the error, for the nth harmonic and a given λ, shows a decreasing trend can be denoted as ${}^{n}k_{crit}$. It can be observed that for $\lambda = 0{\cdot}08$, ${}^{1}k_{crit} = 4$ in the case of the 1st harmonic and the optimum number of terms in the response series should be four. Similarly, three-term series is optimum for representing the 3rd harmonic for $\lambda = 0{\cdot}08$. The approximation errors have been shown as a function of the non-linear parameter λ, for $k = 1$ to 4, in Figures 4(a) and 4(b). The plots in Figures 3 and 4 can be utilized to define a critical value ${}^{n}_{k}\lambda_{crit}$, of the non-dimensional non-linear parameter, to get convergence in

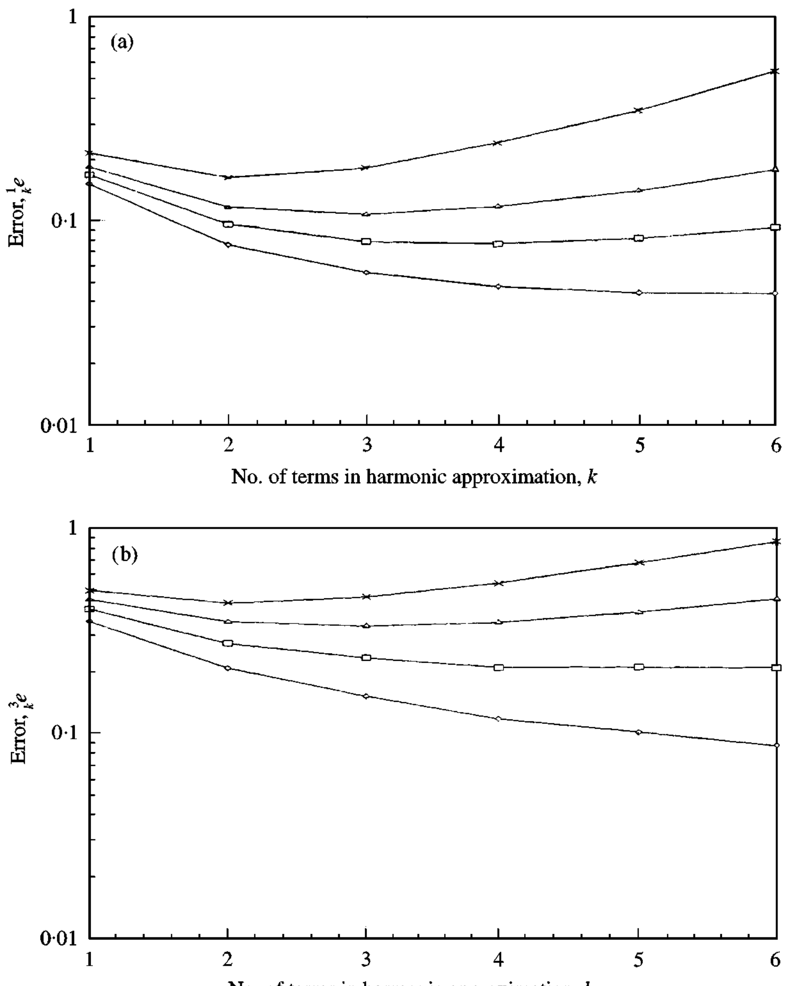

Figure 3. Variation of relative errors with the number of terms, k, in the response harmonic approximation, (for case $r = 0{\cdot}6$): (a) 1st harmonic; —◇—, $\lambda = 0{\cdot}07$; —□—, $\lambda = 0{\cdot}08$; —△—, $\lambda = 0{\cdot}09$; —×—, $\lambda = 0{\cdot}11$. (b) 3rd harmonic; —◇—, $\lambda = 0{\cdot}06$; —□—, $\lambda = 0{\cdot}07$; —△—, $\lambda = 0{\cdot}08$; —×—, $\lambda = 0{\cdot}09$.

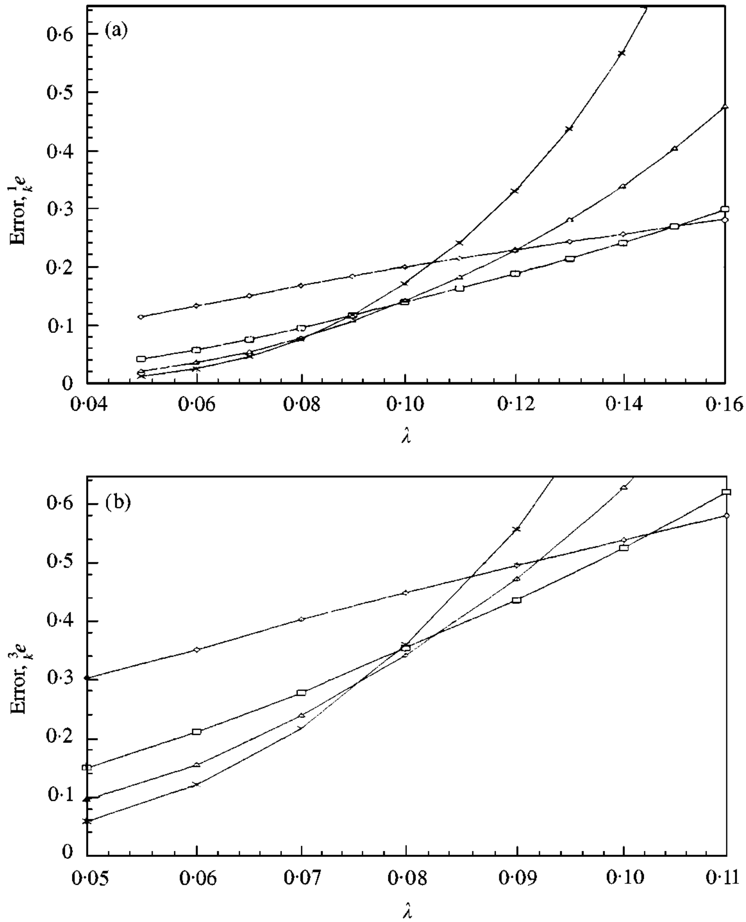

Figure 4. Relative errors in response harmonics for various values of non-dimensional parameter λ, (for case $r = 0{\cdot}6$): (a) 1st harmonic; (b) 3rd harmonic. —◇—, $k = 1$; —□—, $k = 2$; —△—, $k = 3$; —×—, $k = 4$.

a k-term approximation of the nth harmonic. It should be noted that the non-dimensional non-linear parameter λ, involves the non-linear stiffness term k_3 as well as the harmonic force amplitude A, and the critical values ${}^{n}_{k}\lambda_{crit}$, can be suitably employed to decide the excitation levels in experiments. The critical value, ${}^{n}_{k}\lambda_{crit}$, can be defined as the maximum value of λ, for which ${}^{n}_{k}e < {}^{n}_{k-1}e, {}^{n}_{k-2}e, \ldots, {}^{n}_{1}e$. For a four-term approximation of the 1st harmonic, the critical value ${}^{1}_{4}\lambda_{crit}$, of the non-linear parameter is found to be 0·082 (Figure 4(a)), while for a three-term approximation, ${}^{1}_{3}\lambda_{crit}$ is obtained as 0·098 (Figure 4(a)). The critical values ${}^{1}_{k}\lambda_{crit}$ and ${}^{3}_{k}\lambda_{crit}$ (for harmonics $n = 1$ and 3) are plotted in Figures 5(a) and 5(b) respectively, for k ranging from 2 to 7. As shown, these figures help to define the zones of convergence and divergence of a response harmonic as a function of the non-dimensional parameter λ and the number of terms, k, in the approximation. As an example, if the value of the non-dimensional non-linear parameter λ, of a given system is 0·1, then only the first three terms, i.e., $k = 3$, in the approximation, will give a converging solution for the 1st harmonic (Figure 5(a)). For a lower value of $\lambda = 0{\cdot}07$, converged solution is obtained till six terms ($k = 6$) in the approximation. Similar pattern can be observed in the case of the 3rd harmonic, given in Figure 5(b). It is obvious that better accuracy can be obtained with lower

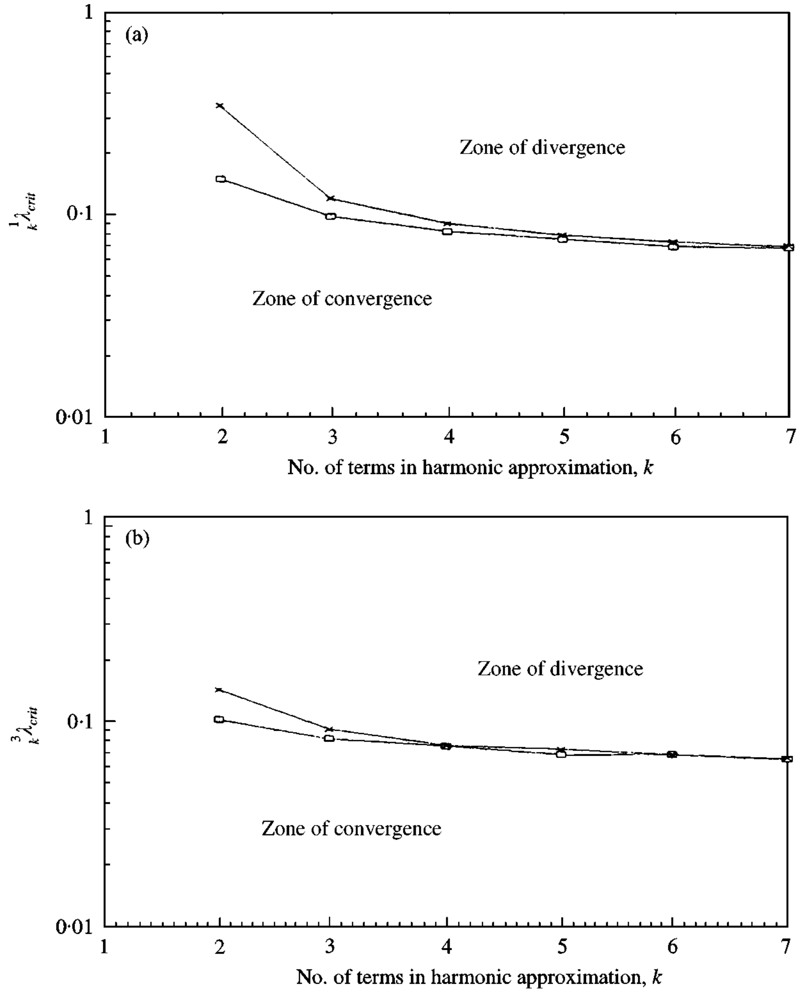

Figure 5. Zones of convergence and divergence, (r = 0·6): (a) 1st harmonic; (b) 3rd harmonic. —□—, error simulation; —×—, ratio test.

values of λ, since more number of terms can be included in the approximation of the response harmonics. Figures 3–5 pertain to a non-dimensional excitation frequency, $r = 0{\cdot}6$. The excitation frequency is varied over a range and the critical values of λ are plotted for k ranging from values 2 to 7, in Figures 6(a)–6(f) for the 1st harmonic and Figures 7(a)–7(c) for the 3rd harmonic. It can be seen that approximations with low values of k either completely miss out on the 1/3 and 1/5 subharmonics at $r = 0{\cdot}33$ and $r = 0{\cdot}2$ respectively, or give non-converging solutions. Critical values $^{n}_{k}\lambda_{crit}$ can be seen to be low at these frequencies. This fact is discussed in the next section.

5. RATIO TEST FOR CONVERGENCE

The procedure for finding the critical value, of the non-dimensional parameter, through numerical simulation, needs iterative computation over a large number of values of λ.

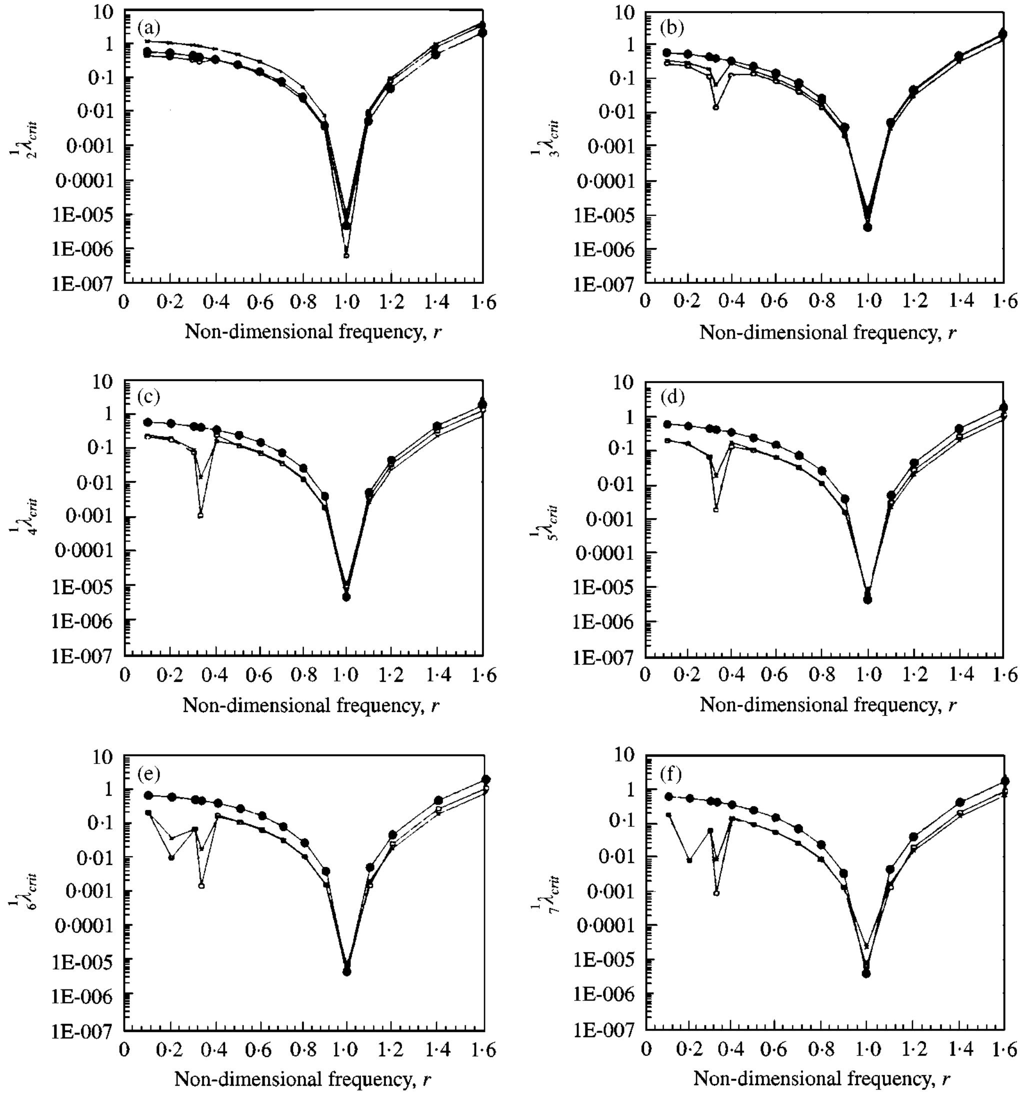

Figure 6. Variation of critical non-dimensional parameter, ${}^{1}_{k}\lambda_{crit}$, with the non-dimensional frequency, r: (a) $k = 2$; (b) $k = 3$; (c) $k = 4$; (d) $k = 5$; (e) $k = 6$; (f) $k = 7$. —□—, error simulation; —×—, ratio test; ●, Tomlinson.

Alternately, ${}^{n}_{k}\lambda_{crit}$ can be determined through application of a simple ratio test to the power series (19). ${}^{n}_{k}\lambda_{crit}$ can be defined, for a k-term approximation of the nth harmonic, as the limiting value of λ, for which all the successive terms, up to k in the approximation, show a decreasing trend, i.e., $\lambda = {}^{n}_{k}\lambda_{crit}$ for the limiting case:

$$\left|\frac{\sigma_k(nr)}{\sigma_{k-1}(nr)}\right| = 1{\cdot}0. \tag{25}$$

Application of the above with equation (19) gives

$$\left|\frac{1}{4}\,\frac{{}^{n+2k-2}C_{k-1}H^{n+k-1,k-1}_{n+2k-2}(r)}{{}^{n+2k-4}C_{k-2}H^{n+k-2,k-2}_{n+2k-4}(r)}\right| = 1 \quad \text{for } \lambda = {}^{n}_{k}\lambda_{crit}. \tag{26}$$

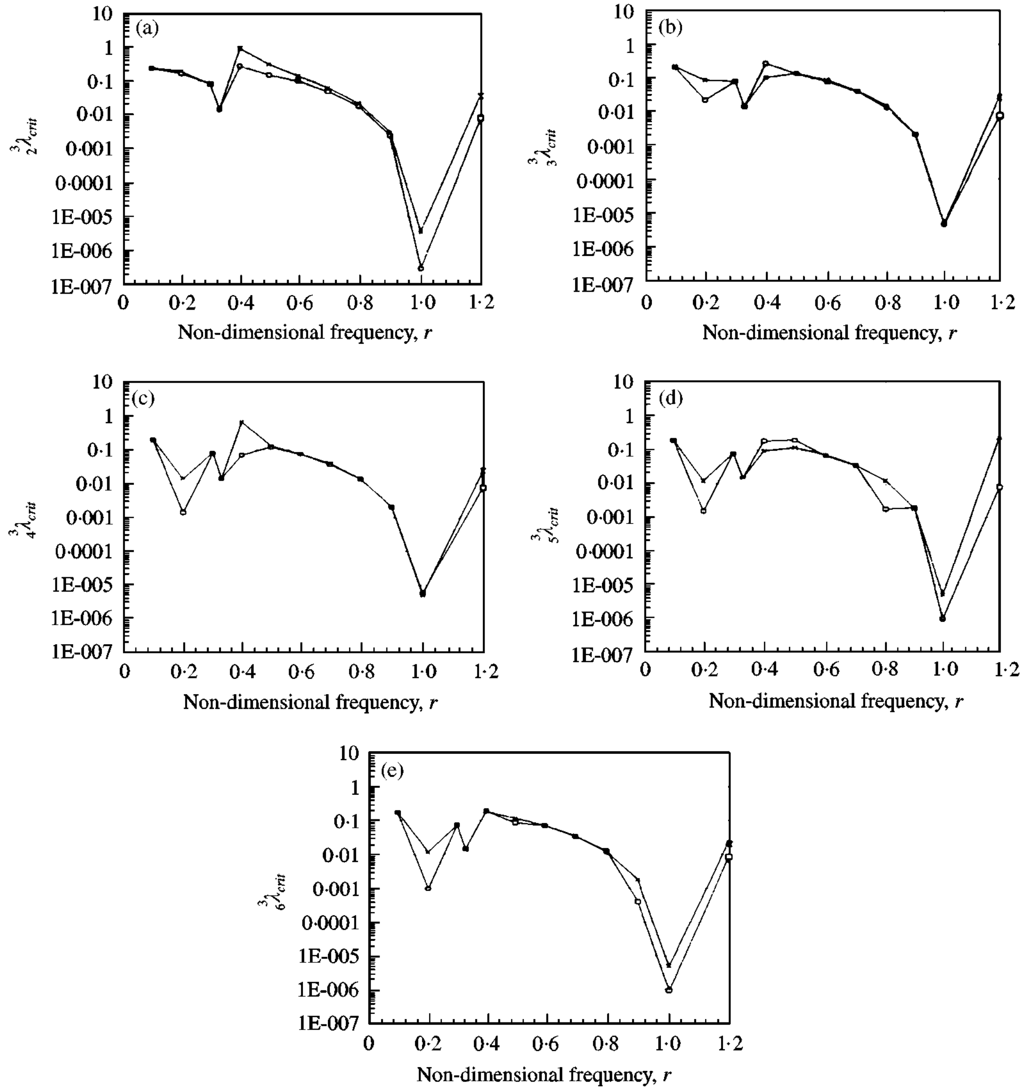

Figure 7. Variation of critical non-dimensional parameter, ${}^{3}_{k}\lambda_{crit}$, with the non-dimensional frequency, r: (a) $k = 2$; (b) $k = 3$; (c) $k = 4$; (d) $k = 5$; (e) $k = 6$. —□—, error simulation; —×—, ratio test.

Employing equation (16b) to express the higher order kernels, in the above, in terms of lower ones, the ratio can be seen to be a function g_k, of the kernel transforms $H_1(r), H_1(3r), H_1(5r), \ldots$, and proportional to λ, i.e.,

$$\left| \frac{1}{4} \frac{{}^{n+2k-2}C_{k-1} H^{n+k-1,k-1}_{n+2k-2}(r)}{{}^{n+2k-4}C_{k-2} H^{n+k-2,k-2}_{n+2k-4}(r)} \right| = \lambda g_k[H_1(r), H_1(3r), H_1(5r), \ldots] \tag{27}$$

providing ${}^{n}_{k}\lambda_{crit}$ as

$${}^{n}_{k}\lambda_{crit} = 1/g_k[H_1(r), H_1(3r), H_1(5r), \ldots]. \tag{28}$$

Using equations (27) and (28), ${}^{n}_{k}\lambda_{crit}$ can be computed as

$$ {}^{n}_{k}\lambda_{crit} = \left| 4\,\frac{{}^{n+2k-4}C_{k-2}\,H^{n+k-2,k-2}_{n+2k-4}(r)}{{}^{n+2k-2}C_{k-1}\,H^{n+k-1,k-1}_{n+2k-2}(r)} \right|_{\lambda=1}. \tag{29} $$

For the 1st and 3rd harmonics, $n = 1$ and $n = 3$, one obtains

$$ {}^{1}_{k}\lambda_{crit} = \left| 4\,\frac{{}^{2k-3}C_{k-2}\,H^{k-1,k-2}_{2k-3}(r)}{{}^{2k-1}C_{k-1}\,H^{k,k-1}_{2k-1}(r)} \right|_{\lambda=1}, \tag{30} $$

$$ {}^{3}_{k}\lambda_{crit} = \left| 4\,\frac{{}^{2k-1}C_{k-2}\,H^{k+1,k-2}_{2k-1}(r)}{{}^{2k+1}C_{k-1}\,H^{k+2,k-1}_{2k+1}(r)} \right|_{\lambda=1} \tag{31} $$

(the subscript on the right-hand side of the above expression denotes that the ratio has been computed at a value of $\lambda = 1$).

The value of ${}^{n}_{k}\lambda_{crit}$ obtained from ratio test is an approximation of its correct value, which is obtained through the error divergence criterion, discussed earlier and illustrated through numerical simulation. However, the approximation of ${}^{n}_{k}\lambda_{crit}$ through the ratio test is fairly good and has been shown, along with the numerically simulated one, in Figures 6(a)–6(f) and 7(a)–7(e) for the 1st and 3rd harmonic respectively. As k increases, the critical value ${}^{n}_{k}\lambda_{crit}$ obtained from ratio test gets closer to the exact one obtained from error simulation. For $k > 3$ ratio test gives very accurate results over a wide range of excitation frequencies. The ratio test also helps to understand the low values of ${}^{n}_{k}\lambda_{crit}$ at the natural frequencies and subharmonics. It can be seen from equation (28) that the ratio $|\sigma_k(nr)/\sigma_{k-1}(nr)|$ is of the order of $\lambda|H_1(nr)|^3$ and the term $|H_1(nr)|$ assumes a large value for $r = 1/n$, making ${}^{n}_{k}\lambda_{crit}$ too small to satisfy convergence criterion given by equation (25). Figures 6 and 7 also show a comparison with the results of Tomlinson [18], who carried out a ratio test on truncated expansion of equation (16b) for higher order kernel transforms. Tomlinson's formula gives reasonable values of the critical parameter only at $r = 1$ and for $k = 2$, i.e., for a two-term series approximation. Away from natural frequency and for $k > 2$ the results deviate considerably from the exact ones, since Tomlinson's formula employs a truncated form of σ_i, by considering their first terms alone. However, subsequent terms of σ_i can be of the same order as its first term, as they may involve kernels of the same order. Also, the error was computed by Tomlinson by averaging over a range of excitation frequencies. Such averaging may entail a large error, since the kernels are frequency sensitive and the critical value of the non-linear parameter λ should be treated as specific to the excitation frequency. Moreover, Tomlinson's analysis was restricted to the first harmonic $Z(r)$ and its convergence was taken to conclude the convergence of the Volterra series.

6. CONVERGENCE FOR A TWO-DEGREE-OF-FREEDOM SYSTEM

The convergence study is extended to the case of a coupled two-degree-of-freedom system with cubic non-linearity in stiffness, given as

$$ m_x\ddot{x} + c_{xx}\dot{x} + k_{xx}x + k_{xy}y + k^N_{xx}x^3 + k^N_{xy}y^3 = f_1(t), $$

$$ m_y\ddot{y} + c_{yy}\dot{y} + k_{yy}y + k_{yx}x + k^N_{yy}y^3 + k^N_{yx}x^3 = f_2(t). \tag{32} $$

Defining non-dimensional parameters

$$\tau = \sqrt{k_{xx}/m_x}\, t, \quad {}^x z(\tau) = x/X_{st}, \quad {}^y z(\tau) = y/X_{st}, \quad X_{st} = f_{\max}/k_{xx},$$

$$\bar{f}_i(\tau) = f_i(\tau)/f_{\max}, \quad i = 1, 2,$$

$$\varsigma_{ii} = \frac{c_{ii}}{2\sqrt{k_{xx} m_x}}, \quad \lambda_{ij}^L = \frac{k_{ij}}{k_{xx}}, \quad \lambda_{ij}^N = \frac{k_{ij}^N f_{\max}^2}{k_{xx}^3}, \quad i = x, y \quad \text{and} \quad j = x, y \quad \mu = m_y/m_x$$

analysis is carried out further with the following non-dimensional equations:

$$\begin{aligned} &{}^x z''(\tau) + 2\varsigma_{xx}\, {}^x z'(\tau) + {}^x z(\tau) + \lambda_{xy}^L\, {}^y z(\tau) + \lambda_{xx}^N\, {}^x z^3(\tau) + \lambda_{xy}^N\, {}^y z^3(\tau) = \bar{f}_1(\tau), \\ &\mu\, {}^y z''(\tau) + 2\varsigma_{yy}\, {}^y z'(\tau) + \lambda_{yy}^L\, {}^y z(\tau) + \lambda_{yx}^L\, {}^x z(\tau) + \lambda_{yy}^N\, {}^y z^3(\tau) + \lambda_{yx}^N\, {}^x z^3(\tau) = \bar{f}_2(\tau). \end{aligned} \tag{33}$$

Volterra series expression for the response ${}^\eta z(\tau)$, η denoting x or y, would be

$$\begin{aligned} {}^\eta z(\tau) = {}^\eta H_0 + \sum_{i=1,2} {}^\eta H_1^{(i)}[\bar{f}_i(\tau)] + \sum_{i=1,2} \sum_{j=1,2} {}^\eta H_2^{(i,j)}[\bar{f}_i(\tau), \bar{f}_j(\tau)] \\ + \sum_{i=1,2} \sum_{j=1,2} \sum_{k=1,2} {}^\eta H_3^{(i,j,k)}[\bar{f}_i(\tau), \bar{f}_j(\tau), \bar{f}_k(\tau)] + \cdots \end{aligned}$$

where

$${}^\eta H_1^{(i)}[\bar{f}_i(\tau)] = \int_{-\infty}^{\infty} {}^\eta h_1^{(i)}(\tau)\, \bar{f}_i(\tau - \tau_1)\, \mathrm{d}\tau_1, \quad i = 1, 2$$

$${}^\eta H_2^{(i,j)}[\bar{f}_i(\tau), \bar{f}_j(\tau)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} {}^\eta h_2^{(i,j)}(\tau)\, \bar{f}_i(\tau - \tau_1)\, \bar{f}_j(\tau - \tau_2)\, \mathrm{d}\tau_1\, \mathrm{d}\tau_2,$$

for $i = 1, 2$ and $j = 1, 2$,

$$\begin{aligned} &{}^\eta H_3^{(i,j,k)}[\bar{f}_i(\tau), \bar{f}_j(\tau)\, \bar{f}_k(\tau)] \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} {}^\eta h_3^{(i,j,k)}(\tau)\, \bar{f}_i(\tau - \tau_1)\, \bar{f}_j(\tau - \tau_2)\, \bar{f}_k(\tau - \tau_3)\, \mathrm{d}\tau_1\, \mathrm{d}\tau_2\, \mathrm{d}\tau_3 \end{aligned}$$

for $i = 1, 2$, $j = 1, 2$ and $k = 1, 2$.

For a single-tone non-dimensional harmonic excitation

$$\bar{f}_1(\tau) = \mathrm{e}^{\mathrm{j}r\tau}/2 + \mathrm{e}^{-\mathrm{j}r\tau}/2; \qquad \bar{f}_2(\tau) = 0 \quad \text{with } r = \omega/\sqrt{k_{xx}/m_x}$$

the response is

$${}^\eta z(\tau) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \sum_{p+q=n} {}^n C_q\, {}^\eta H_n^{p,q}(r)\, \mathrm{e}^{\mathrm{j}r_{p,q}\tau}. \tag{34}$$

Substituting equation (34) into equation (33) gives

$$\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n \sum_{p+q=n} {}^nC_q\, {}^xH_n^{p,q}(r)\, \mathrm{e}^{\mathrm{j}r_{p,q}\tau}[-r_{p,q}^2+1+\mathrm{j}2\varsigma_{xx}r_{p,q}]$$

$$+\lambda_{xy}^L\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n \sum_{p+q=n} {}^nC_q\, {}^yH_n^{p,q}(r)\, \mathrm{e}^{\mathrm{j}r_{p,q}\tau}$$

$$+\lambda_{xx}^N\left[\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n \sum_{p+q=n} {}^nC_q\, {}^xH_n^{p,q}(r)\, \mathrm{e}^{\mathrm{j}r_{p,q}\tau}\right]^3$$

$$+\lambda_{xy}^N\left[\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n \sum_{p+q=n} {}^nC_q\, {}^yH_n^{p,q}(r)\, \mathrm{e}^{\mathrm{j}r_{p,q}\tau}\right]^3=\frac{\mathrm{e}^{\mathrm{j}r\tau}}{2}+\frac{\mathrm{e}^{-\mathrm{j}r\tau}}{2}, \tag{35}$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n \sum_{p+q=n} {}^nC_q\, {}^yH_n^{p,q}(r)\, \mathrm{e}^{\mathrm{j}r_{p,q}\tau}[-\mu r_{p,q}^2+1+\mathrm{j}2\varsigma_{yy}r_{p,q}]$$

$$+\lambda_{yx}^L\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n \sum_{p+q=n} {}^nC_q\, {}^xH_n^{p,q}(r)\, \mathrm{e}^{\mathrm{j}r_{p,q}\tau}$$

$$+\lambda_{yy}^N\left[\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n \sum_{p+q=n} {}^nC_q\, {}^yH_n^{p,q}(r)\, \mathrm{e}^{\mathrm{j}r_{p,q}\tau}\right]^3$$

$$+\lambda_{yx}^N\left[\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n \sum_{p+q=n} {}^nC_q\, {}^xH_n^{p,q}(r)\, \mathrm{e}^{\mathrm{j}r_{p,q}\tau}\right]^3=0. \tag{36}$$

Kernel transforms expressions are obtained by equating the coefficients of $(1/2)^n\,\mathrm{e}^{\mathrm{j}r_{p,q}\tau}$.

For $n=1$, one gets

$$[-r_{1,0}^2+1+\mathrm{j}2\varsigma_{xx}r_{1,0}]\,{}^xH_1^{1,0}(r)+\lambda_{xy}^L\,{}^yH_1^{1,0}(r)=1,$$

$$[-\mu r_{1,0}^2+\lambda_{yy}^L+\mathrm{j}2\varsigma_{yy}r_{1,0}]\,{}^yH_1^{1,0}(r)+\lambda_{yx}^L\,{}^xH_1^{1,0}(r)=0. \tag{37}$$

Noting that $r_{1,0}=r$, ${}^xH_1^{1,0}(r)={}^xH_1(r)$ and ${}^yH_1^{1,0}(r)={}^yH_1(r)$, the above is simplified as

$$[-r^2+1+\mathrm{j}2\varsigma_{xx}r]\,{}^xH_1(r)+\lambda_{xy}^L\,{}^yH_1(r)=1,$$

$$\lambda_{yx}^L\,{}^xH_1(r)+[-\mu r^2+\lambda_{yy}^L+\mathrm{j}2\varsigma_{yy}r]\,{}^yH_1(r)=0, \tag{38}$$

which provide the definitions of the following first-order transforms:

$${}^xH_1(r)=\frac{[-\mu r^2+\lambda_{yy}^L+\mathrm{j}2\varsigma_{yy}r]}{[-r^2+1+\mathrm{j}2\varsigma_{xx}r][-\mu r^2+\lambda_{yy}^L+\mathrm{j}2\varsigma_{yy}r]-\lambda_{xy}^L\lambda_{yx}^L},$$

$${}^yH_1(r)=\frac{-\lambda_{yx}^L}{[-r^2+1+\mathrm{j}2\varsigma_{xx}r][-\mu r^2+\lambda_{yy}^L+\mathrm{j}2\varsigma_{yy}r]-\lambda_{xy}^L\lambda_{yx}^L}.$$

Equating the coefficients of $(1/2)^n \mathrm{e}^{\mathrm{j}r_{p,q}\tau}$, for $n > 1$, gives

$$[-r_{p,q}^2 + 1 + \mathrm{j}2\varsigma_{xx} r_{p,q}]\,{}^nC_q\,{}^xH_n^{p,q}(r) + \lambda_{xy}^L\,{}^nC_q\,{}^yH_n^{p,q}(r)$$

$$+ \lambda_{xx}^N \sum_{\substack{p_i+q_i=n_i\\ n_1+n_2+n_3=n}} [{}^{n_1}C_{q_1}\,{}^xH_{n_1}^{p_1,q_1}(r)]\,[{}^{n_2}C_{q_2}\,{}^xH_{n_2}^{p_2,q_2}(r)]\,[{}^{n_3}C_{q_3}\,{}^xH_{n_3}^{p_3,q_3}(r)]$$

$$+ \lambda_{xy}^N \sum_{\substack{p_i+q_i=n_i\\ n_1+n_2+n_3=n}} [{}^{n_1}C_{q_1}\,{}^yH_{n_1}^{p_1,q_1}(r)]\,[{}^{n_2}C_{q_2}\,{}^yH_{n_2}^{p_2,q_2}(r)]\,[{}^{n_3}C_{q_3}\,{}^yH_{n_3}^{p_3,q_3}(r)] = 0. \quad (39)$$

and

$$[-\mu r_{p,q}^2 + \lambda_{yy}^L + \mathrm{j}2\varsigma_{yy} r_{p,q}]\,{}^nC_q\,{}^yH_n^{p,q}(r) + \lambda_{yx}^L\,{}^nC_q\,{}^xH_n^{p,q}(r)$$

$$+ \lambda_{yx}^N \sum_{\substack{p_i+q_i=n_i\\ n_1+n_2+n_3=n}} [{}^{n_1}C_{q_1}\,{}^xH_{n_1}^{p_1,q_1}(r)]\,[{}^{n_2}C_{q_2}\,{}^xH_{n_2}^{p_2,q_2}(r)]\,[{}^{n_3}C_{q_3}\,{}^xH_{n_3}^{p_3,q_3}(r)]$$

$$+ \lambda_{yy}^N \sum_{\substack{p_i+q_i=n_i\\ n_1+n_2+n_3=n}} [{}^{n_1}C_{q_1}\,{}^yH_{n_1}^{p_1,q_1}(r)]\,[{}^{n_2}C_{q_2}\,{}^yH_{n_2}^{p_2,q_2}(r)]\,[{}^{n_3}C_{q_3}\,{}^yH_{n_3}^{p_3,q_3}(r)] = 0. \quad (40)$$

Representing

$$B_1 = \lambda_{xx}^N \sum_{\substack{p_i+q_i=n_i\\ n_1+n_2+n_3=n}} [{}^{n_1}C_{q_1}\,{}^xH_{n_1}^{p_1,q_1}(r)]\,[{}^{n_2}C_{q_2}\,{}^xH_{n_2}^{p_2,q_2}(r)]\,[{}^{n_3}C_{q_3}\,{}^xH_{n_3}^{p_3,q_3}(r)]$$

$$+ \lambda_{xy}^N \sum_{\substack{p_i+q_i=n_i\\ n_1+n_2+n_3=n}} [{}^{n_1}C_{q_1}\,{}^yH_{n_1}^{p_1,q_1}(r)]\,[{}^{n_2}C_{q_2}\,{}^yH_{n_2}^{p_2,q_2}(r)]\,[{}^{n_3}C_{q_3}\,{}^yH_{n_3}^{p_3,q_3}(r)]$$

and

$$B_2 = \delta\lambda_{yx}^N \sum_{\substack{p_i+q_i=n_i\\ n_1+n_2+n_3=n}} [{}^{n_1}C_{q_1}\,{}^xH_{n_1}^{p_1,q_1}(r)]\,[{}^{n_2}C_{q_2}\,{}^xH_{n_2}^{p_2,q_2}(r)]\,[{}^{n_3}C_{q_3}\,{}^xH_{n_3}^{p_3,q_3}(r)]$$

$$+ \lambda_{yy}^N \sum_{\substack{p_i+q_i=n_i\\ n_1+n_2+n_3=n}} [{}^{n_1}C_{q_1}\,{}^yH_{n_1}^{p_1,q_1}(r)]\,[{}^{n_2}C_{q_2}\,{}^yH_{n_2}^{p_2,q_2}(r)]\,[{}^{n_3}C_{q_3}\,{}^yH_{n_3}^{p_3,q_3}(r)],$$

solutions of higher order kernel transforms can be obtained as

$${}^xH_n^{p,q}(r) = \frac{-B_1[-\mu r_{p,q}^2 + \lambda_{yy}^L + \mathrm{j}2\varsigma_{yy} r_{p,q}] + B_2\lambda_{xy}^L}{{}^nC_q\{[-r_{p,q}^2 + 1 + \mathrm{j}2\varsigma_{xx} r_{p,q}][-\mu r_{p,q}^2 + \lambda_{yy}^L + \mathrm{j}2\varsigma_{yy} r_{p,q}] - \lambda_{xy}^L\lambda_{yx}^L\}}, \quad (41)$$

$${}^yH_n^{p,q}(r) = \frac{-B_2[-r_{p,q}^2 + 1 + \mathrm{j}2\varsigma_{xx} r_{p,q}] + B_1\lambda_{yx}^L}{{}^nC_q\{[-r_{p,q}^2 + 1 + \mathrm{j}2\varsigma_{xx} r_{p,q}][-\mu r_{p,q}^2 + \lambda_{yy}^L + \mathrm{j}2\varsigma_{yy} r_{p,q}] - \lambda_{xy}^L\lambda_{yx}^L\}}. \quad (42)$$

Response series (34) can be rearranged, similar to equation (17), in terms of frequency components as

$$ {}^{\eta}z(\tau) = |{}^{\eta}Z(r)|\cos(r\tau + {}^{\eta}\psi_1) + |{}^{\eta}Z(3r)|\cos(3r\tau + {}^{\eta}\psi_3) + \cdots, \qquad (43) $$

where

$$ {}^{\eta}Z(nr) = 2\sum_{i=1}^{\infty}\left(\frac{1}{2}\right)^{n}\ {}^{n+2i-2}C_{i-1}\ {}^{\eta}H_{n+2i-2}^{n+i-1,i-1}(r), \quad {}^{\eta}\psi_n = \angle{}^{\eta}Z(nr), $$

$$ \eta = x \text{ or } y \quad \text{and} \quad n = 1, 3, 5, \ldots $$

A k-term approximation of the response series would be given by

$$ {}^{\eta}_{k}Z(nr) = \sum_{i=1}^{k} {}^{\eta}\sigma_i(nr), \qquad (44) $$

where

$$ {}^{\eta}\sigma_i(nr) = 2\left(\frac{1}{2}\right)^{n}\ {}^{n+2i-2}C_{i-1}\ {}^{\eta}H_{n+2i-2}^{n+i-1,i-1}(r). \qquad (45) $$

The error between above approximation and the exact value of the harmonic is

$$ {}^{\eta}e_k(nr) = |[{}^{\eta}Z(nr) - {}^{\eta}_{k}Z(nr)]/{}^{\eta}Z(nr)|. $$

The approximated response harmonic ${}^{\eta}_{k}Z(nr)$ can be synthesized after determining the higher order kernel transforms ${}^{\eta}H_{n+2i-2}^{n+i-1,i-1}(r)$, using equations (41)–(45). The exact response harmonics are obtained by fourth order Runge–Kutta numerical solution of equation (33). The limit of convergence of the response harmonics ${}^{\eta}_{k}Z(nr)$ is represented by the set of critical values of the non-dimensional parameters ${}^{\eta}_{k}(\lambda_{xx}^{N}, \lambda_{yy}^{N}, \lambda_{xy}^{N}, \lambda_{yx}^{N})_{crit}^{\eta}$. The four non-dimensional non-linear parameters along with the number of terms k constitute a five-dimensional parametric space which is divided into two regions of convergence and divergence by the hypersurface represented by all possible sets of ${}^{\eta}_{k}(\lambda_{xx}^{N}, \lambda_{yy}^{N}, \lambda_{xy}^{N}, \lambda_{yx}^{N})_{crit}^{\eta}$, for a given k. For simplicity, cross non-linear stiffnesses, λ_{xy}^{N} and λ_{yx}^{N}, are taken as zero in the present computer simulation. The parametric space, then reduces to three-dimensional space comprising of λ_{xx}^{N}, λ_{yy}^{N} and k.

The following values are considered for computer simulation of equations (33):

$$ \mu = 1{\cdot}0, \quad \lambda_{xx}^{L} = \lambda_{yy}^{L} = 1{\cdot}0, \quad \lambda_{xy}^{L} = \lambda_{xy}^{L} = 0{\cdot}1 \text{ and } 0{\cdot}5, \quad \varsigma_{xx} = \varsigma_{yy} = 0{\cdot}01. $$

Defining a scaling parameter $\beta = \lambda_{yy}^{N}/\lambda_{xx}^{N}$, the convergence is analyzed numerically as a function of λ_{xx}^{N}, β and the number of terms, k, in the response harmonic series. For a specific value of β, the limiting value of λ_{xx}^{N} for which a k-term series is convergent is termed as the critical non-dimensional parameter ${}^{\eta}_{k}\lambda_{crit}^{(\eta)}$. The variation of this parameter with the excitation frequency r, for a typical value of $k = 3$, is shown for both x and y direction response in Figures 8(a) and 8(b). The plots are shown for three different values of the scaling parameter $\beta = 0{\cdot}1, 1{\cdot}0, 10{\cdot}0$. These plots pertain to a case of linear coupling numerically taken as $\lambda_{xy}^{L} = \lambda_{yx}^{L} = 0{\cdot}5$. It can be observed that the critical values ${}^{\eta}_{k}\lambda_{crit}^{(\eta)}$ are small at the system natural frequencies (which for $\lambda_{xy}^{L} = \lambda_{yx}^{L} = 0{\cdot}5$ are $r = 0{\cdot}7$ and $r = 1{\cdot}224$) and their 1/3 subharmonics at $r = 0{\cdot}233$ and $0{\cdot}408$. It is maximum in the vicinity of the anti-resonance frequency at $r = 1{\cdot}0$. Similar characteristics are observed, for a case with

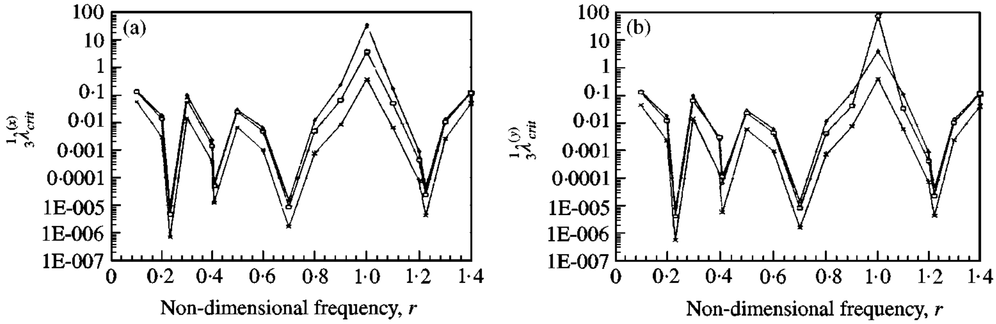

Figure 8. Variation of critical non-dimensional parameter, ${}^{1}_{3}\lambda_{crit}$, with non-dimensional Frequency, r, for two-degree-of-freedom system ($\lambda^L_{xy} = \lambda^L_{yx} = 0{\cdot}5$): (a) x-response; (b) y-response. —◇—, $\beta = 0{\cdot}1$; —□—, $\beta = 1{\cdot}0$; —×—, $\beta = 10{\cdot}0$.

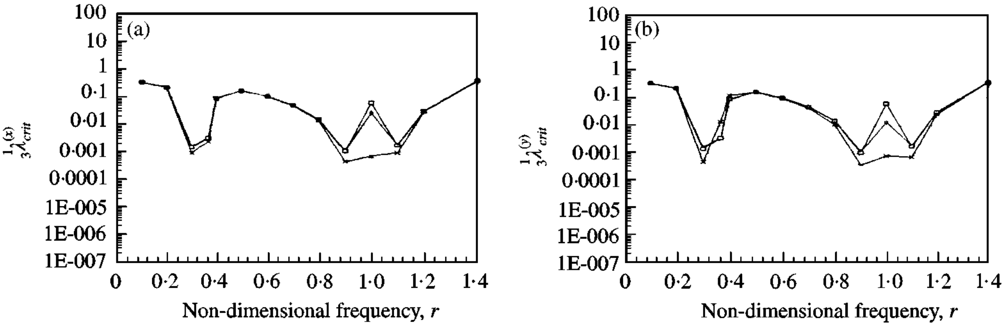

Figure 9. Variation of critical non-dimensional parameter, ${}^{1}_{3}\lambda_{crit}$, with non-dimensional Frequency, r, for two-degree-of-freedom system ($\lambda^L_{xy} = \lambda^L_{yx} = 0{\cdot}1$): (a) x-response; (b) y-response. —◇—, $\beta = 0{\cdot}1$; —□—, $\beta = 1{\cdot}0$; —×—, $\beta = 10{\cdot}0$.

weaker linear coupling ($\lambda^L_{xy} = \lambda^L_{yx} = 0{\cdot}1$), in Figures 9(a) and 9(b). However, the coupling being weak, the effect of scaling parameter β is also weaker than in the previous case, of Figures 8(a) and 8(b).

The critical values ${}^{n}_{k}\lambda^{(\eta)}_{crit}$ discussed above are obtained through the ratio test similar to that of single-degree-of-freedom system. A comparison of these critical values with those obtained through iterative simulation is shown in Figures 10(a)–10(c) for various values of scaling parameter β. The values agree reasonably well for a series with three terms or more.

7. CONCLUSIONS

Convergence limitations of Volterra series expression of non-linear system response have been studied for Duffing oscillator under harmonic excitation. The convergence is found to be a function of the non-dimensional non-linear parameter and also dependent on the number of terms considered in the response series. Convergence threshold of the series representation has been defined in terms of a critical non-dimensional parameter and an algorithm based on ratio test has been presented to determine the critical value of the non-dimensional parameter. The suggested procedure gives accurate estimation of the critical value over a wide range of excitation frequencies. The method has been extended for

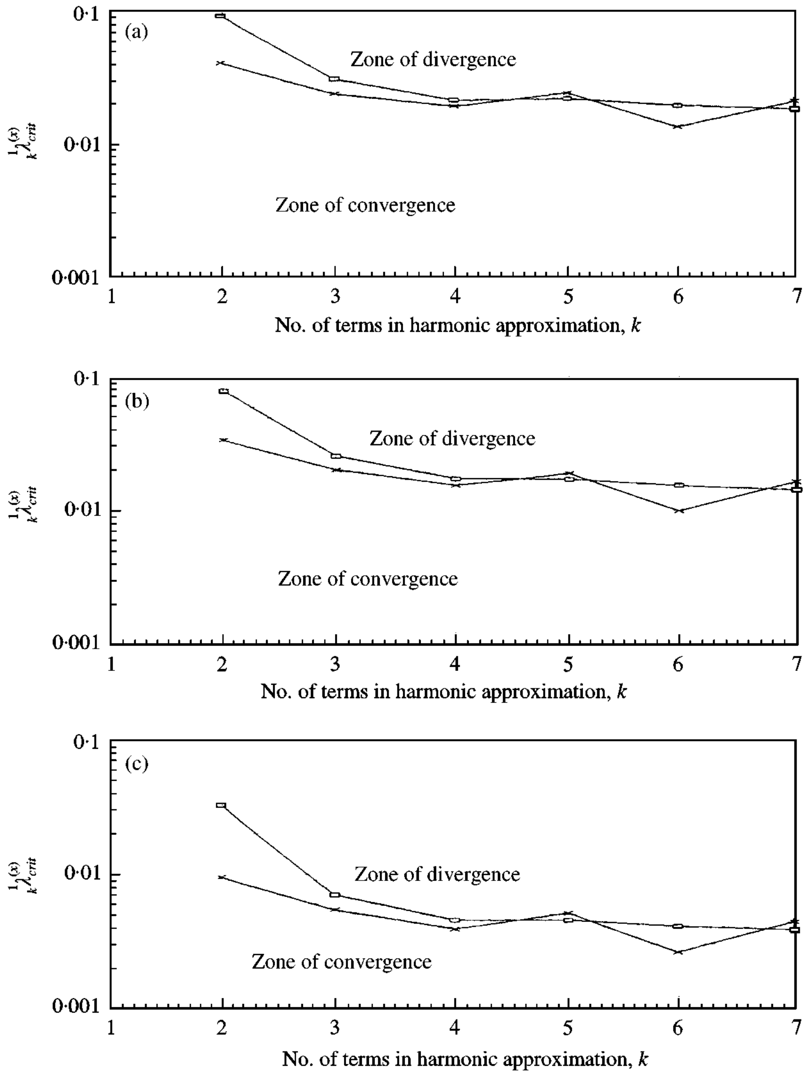

Figure 10. Convergence-divergence zone of two-degree-of-freedom system (x response, $r = 0{\cdot}5$, $\lambda^L_{xy} = \lambda^L_{yx} = 0{\cdot}5$): (a) $\beta = 0{\cdot}1$; (b) $\beta = 1{\cdot}0$; (c) $\beta = 10{\cdot}0$. —□—, ratio test; —×—, error simulation.

a two-degree-of-freedom system. The critical value of non-dimensional parameter in this case is found to be dependent on the non-linear stiffness coefficients as well as on the linear coupling stiffness coefficients. The suggested method can be employed to design experiments and set the limiting value of harmonic excitation level for a certain number of terms in the response series.

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REFERENCES

1. V. Volterra 1958 *Theory of Functionals and of Integral and Integro-differential Equations*. New York: Dover Publications, Inc.
2. M. Schetzen 1980 *The Volterra and Wiener Theories of Nonlinear Systems*. New York: John Wiley-Interscience.
3. S. A. Billings 1980 *Proceedings of IEEE* **127**, 272–285. Identification of non-linear systems—a survey.
4. M. J. Korenberg and I. W. Hunter 1990 *Annals of Biomedical Engineering* **18,** 629–654. The identification of non-linear biological systems: Wiener kernel approach.
5. W. J. Rugh 1981 *Nonlinear System Theory*: *The Volterra/Wiener Approach*. Baltimore, MD: John Hopkins University Press.
6. N. Wiener 1958 *Nonlinear Problems in Random Theory*. New York: Wiley.
7. Y. W. Lee and M. Schetzen 1965 *International Journal of Control* **2**, 237–254. Measurement of the Wiener kernels of a non-linear system by cross-correlation.
8. E. Bedrosian and S. O. Rice 1971 *Proceedings of IEEE* **59**, 1688–1707. The output properties of Volterra systems driven by harmonic and Gaussian inputs.
9. I. I. Orabi and G. Ahmadi 1987 *International Journal of Nonlinear Mechanics* **22**, 451–465. A functional series expansion method for response analysis of non-linear systems subjected to random excitations.
10. S. J. Gifford and G. R. Tomlinson 1989 *Journal of Sound and Vibration* **135**, 289–317. Recent advances in the application of functional series to non-linear structures.
11. J. S. Bendat 1990 *Nonlinear System Analysis and Identification from Random Data*. New York: John Wiley & Sons.
12. A. A. Khan and N. S. Vyas 1999 *Journal of Sound and Vibration* **221**, 805–821. Non-linear parameter estimation using Volterra and Wiener theories.
13. G. S. Christensen 1968 *IEEE Transaction on Automatic Control* **13**, 736–737. On the convergence of Volterra series.
14. A. Czarniak and J. Kudrewicz 1984 *IEEE Transaction on Circuits and Systems* **CAS-31**, 751–752. The convergence of Volterra series for non-linear networks.
15. I. W. Sandberg 1992 *IEEE Transaction on Signal Processing* **40**, 1438–1442. Uniform approximation with doubly finite Volterra series.
16. S. Boyd, Y. S. Tang and L. O. Chua 1983 *IEEE Transaction on Circuits and Systems* **CAS-30**, 571–577. Measuring Volterra kernels.
17. A. Ushida and L. O. Chua 1984 *IEEE Transaction On Circuits and Systems* **CAS-31**, 766–778. Frequency-domain analysis of non-linear circuits driven by multi-tone signals.
18. G. R. Tomlinson and G. Manson 1996 *Journal of Sound and Vibration* **190**, 751–762. A simple criterion for establishing an upper limit to the harmonic excitation level of the Duffing oscillator using the Volterra series.